

## FINDING ALL MONOMIALS IN A POLYNOMIAL IDEAL

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**ABSTRACT.** Given a  $d \times n$  integer matrix  $A$ , the main result is an elementary, simple-to-state algorithm that finds the largest  $A$ -graded ideal contained in any ideal  $I$  in a polynomial ring  $\mathbb{k}[\mathbf{x}]$  in  $n$  variables. The special case where  $A$  is an identity matrix yields that  $(\mathbf{t}.I) \cap \mathbb{k}[\mathbf{x}]$  is the largest monomial ideal in  $I$ , where the generators of  $\mathbf{t}.I$  are those of  $I$  but with each variable  $x_i$  replaced by  $t_i x_i$  for an invertible variable  $t_i$ .

It is easy to tell whether an ideal  $I$  in a polynomial ring  $\mathbb{k}[\mathbf{x}] = \mathbb{k}[x_1, \dots, x_n]$  contains at least one monomial: it does so if and only if the saturation  $(I : (x_1 \cdots x_n)^\infty)$  is the unit ideal. Being more precise about the monomials in  $I$  makes the problem a little harder. Here are three equivalent ways to formulate it.

**Question 1.** Fix an ideal  $I \subseteq \mathbb{k}[\mathbf{x}]$ .

1. What is the set of monomials in  $I$ ?
2. What is the largest  $\mathbb{N}^n$ -graded ideal contained in  $I$ ?
3. What is the smallest  $(\mathbb{k}^*)^n$ -scheme containing the zero scheme of  $I$ ?

**Answer 2.** Let  $\mathbf{t} = t_1, \dots, t_n$  be a new set of variables. Inside of the Laurent polynomial ring  $\mathbb{k}[\mathbf{x}][\mathbf{t}^{\pm 1}]$ , let  $\mathbf{t}.I$  denote the ideal whose generators are those of  $I$  where each variable  $x_i$  is replaced by  $t_i x_i$ . The biggest monomial ideal contained in  $I$  is  $(\mathbf{t}.I) \cap \mathbb{k}[\mathbf{x}]$ .

This answer appears, with non-invertible  $\mathbf{t}$ -variables, as Algorithm 4.4.2 in [SST00]. An elementary proof is given there. It is obvious, for instance, that every monomial in  $I$  lies in  $(\mathbf{t}.I) \cap \mathbb{k}[\mathbf{x}]$ , since the  $\mathbf{t}$  variables are units; and intuitively, there is no way to clear all of the  $\mathbf{t}$  variables simultaneously from all of the monomials in a given polynomial with more than one term. That said, viewing Question 1 as a special case of a more general problem from multigraded algebra lends insight. For notation, if  $A \in \mathbb{Z}^{d \times n}$  is a  $d \times n$  matrix of integers, to say that the polynomial ring  $\mathbb{k}[\mathbf{x}]$  is  $A$ -graded means that each monomial  $\mathbf{x}^{\mathbf{b}} \in \mathbb{k}[\mathbf{x}]$  is assigned the  $A$ -degree  $\deg(\mathbf{x}^{\mathbf{b}}) = A\mathbf{b}$ , the linear combination of the  $n$  columns of the matrix  $A$  with coefficients  $\mathbf{b} = b_1, \dots, b_n$ . An ideal  $I$  is  $A$ -graded if it is generated by polynomials whose terms all have the same  $A$ -degree.

**Theorem 3.** Fix an ideal  $I \subseteq \mathbb{k}[\mathbf{x}]$  and a  $d \times n$  matrix  $A$  with column vectors  $\mathbf{a}_1, \dots, \mathbf{a}_n$ . Let  $\mathbf{t} = t_1, \dots, t_n$  be a new set of variables. Denote by  $\mathbf{t}.I \subseteq \mathbb{k}[\mathbf{x}][\mathbf{t}^{\pm 1}]$  the ideal whose generators are those of  $I$  with each variable  $x_i$  replaced by  $\mathbf{t}^{\mathbf{a}_i} x_i$ . The largest  $A$ -graded ideal contained in  $I$  equals the intersection  $(\mathbf{t}.I) \cap \mathbb{k}[\mathbf{x}]$ .

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After the first version of this note was posted, the authors of [KR05] pointed out that the statement of Theorem 3 is essentially Tutorial 50(a) in their book, an exercise with a suggested proof that is different from the one here.

**Remark 4.** Details on  $A$ -graded algebra in general can be found in [MS05, Chapter 8]. The  $A$ -grading on  $\mathbb{k}[\mathbf{x}]$  corresponds uniquely to the action of a torus  $T \cong (\mathbb{k}^*)^d$  on  $\mathbb{k}^n$ . (References for this are hard to locate. An exposition appears in Appendix A.1 of the first arXiv version of [KM05], at <http://arxiv.org/abs/math/0110058v1>.) Under this correspondence,  $A$ -graded ideals correspond to subschemes of  $\mathbb{k}^n$  that carry  $T$ -actions. Therefore the zero scheme of the ideal  $(\mathbf{t}.I) \cap \mathbb{k}[\mathbf{x}]$  in Theorem 3 is the smallest  $T$ -scheme containing the zero scheme  $Z(I)$ .

*Proof of Theorem 3.* Let  $X = T \times \mathbb{k}^n$ . Create a subbundle  $Y \subseteq X$  over  $T$  whose fiber over  $\tau \in T$  is the translate  $\tau^{-1}.Z(I)$  of the zero-scheme  $Z(I)$  by  $\tau^{-1}$ . The image of the projection of  $Y$  to  $\mathbb{k}^n$  is the minimal  $T$ -stable scheme containing  $Z(I)$  by construction: it is the union of all  $T$ -translates of  $Z(I)$ . Therefore the vanishing ideal of the image of the projection is the maximal  $A$ -graded subideal of  $I$ . The scheme  $Y$  is expressed, in coordinates, as the zero scheme of  $\mathbf{t}.I$ , and the image of its projection to  $\mathbb{k}^n$  is the zero scheme of  $(\mathbf{t}.I) \cap \mathbb{k}[\mathbf{x}]$ .  $\square$

**Remark 5.** In contrast to the monomial situation, the binomial analogue of Question 1.1, which begins with, “Is there a binomial in  $I$ ?”, appears to be much harder than the monomial question, as observed by Jensen, Kahle, and Katthän [JKK16]. They note, for example, that for each  $d$  there is an ideal in  $\mathbb{k}[x, y]$  that contains no binomials of degree less than  $d$  but nonetheless has a quadratic Gröbner basis and contains a binomial of degree  $d$ .

## REFERENCES

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